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Mixed boundary-value problems of two-dimensional anisotropic thermoelasticity with elliptic boundaries

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Abstract

The two-dimensional mixed boundary-value problems for an anisotropic thermoelastic body containing an elliptic hole boundary are considered in this paper. By using the formalism of Stroh [Phil. Mag. 7, 625–646], the approach of analytic function continuation and the technique of conformal mapping, a unified analytical solution for elliptic hole boundaries and for general anisotropic thermoelastic media is provided. As an application, two typical examples associated with mixed boundary-value problems are solved completely. One is an indentation problem over an elliptic hole boundary, the other is a partially reinforced elliptic hole under a remote uniform heat flow. Both the contact stress under the rigid stamp and the bonded stress along the reinforced segment are studied in detail and shown in graphic form. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords: Analytic function continuation; Conformal mapping; Elliptic hole boundary

1. Introduction

In formulating the mixed boundary-value problems in continuum mechanics, the boundary conditions usually consist of two types. One is the potential type such as temperature, velocity potential, electrostatic potential and displacement; the other is the flux type such as heat flux, velocity, electrostatic charge and stress. Accordingly, for elasticity problems in solid mechanics, one may classify the displacements and the stresses as respectively the potential and flux type quantities. The physical of the problem requires the potential quantities be bounded but allows the flux quantities to be unbounded which may be named as flux singularity. If the type of boundary conditions changes at a discontinuous point of the boundary, the resulting strength of singularity at this point will be significantly enhanced due to the presence of both geometric and flux singularities. This is the reason why the mixed boundary-value problems are more difficult to solve than the ordinary boundary-value problems such as the traction- and displacement-boundary-value problems.

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Mixed boundary-value problems in two-dimensional elasticity such as punch problems and interface crack problems have been solved and collected by Muskhelishvili (1953) and England (1971) for isotropic materials. In solving mixed boundary-value problems the analytical function approach is found to be a simpler and powerful procedure. As to anisotropic materials, the interface crack problems were investigated by Ting (1986), Qu and Bassani (1989), Suo (1990), and Gao et al. (1992); and the punch problems for an elastic half-plane were solved by Fan and Keer (1994) and Fan and Hwu (1996). Among them, the Stroh formalism (Stroh, 1958) and the analytic function approach have been utilized and found to be the most appropriate for solving mixed boundary-value problem of two-dimensional anisotropic elasticity. Through these methods, many exact closed-form solutions have been obtained for the elasticity problems with simpler geometry such as straight boundaries. Most of important practical applications of elasticity theory are concerned with solids which have curvilinear boundaries. However, due to the mathematical infeasibility, the corresponding problems associated with curvilinear boundaries have received less attention in the literature. Recently, a general solution for the problems of rigid stamp indentation on a curvilinear hole boundary of an anisotropic elastic body was given by Fan and Hwu (1998). To solve the problems with curvilinear boundaries, a one-to-one mapping needs to transform an awkwardly shaped region to a simple one. However, there are certain cases that a one-to-one mapping function cannot be found. For example, there is no exact closed-form solution for the generally anisotropic plate containing a polygonal hole for which the mapping function is not single-valued. This is due to the difficulty of finding the mapping functions so that the three image points associated with three different eigenvalues for anisotropic elasticity on the unit circle in the transformed domain are always coincident. The only exception occurs when the hole boundary is an ellipse in which the mapping functions can be given explicitly so that any point on the hole boundary under three different mappings is mapped onto a single point on the unit circle in the transformed domain.

In this paper, we like to deal with the mixed boundary-value problems of two-dimensional anisotropic thermoelasticity with elliptic boundaries. Based on the Stroh formalism (Stroh, 1958), the method of analytical continuation, and the technique of conformal mapping, a unified analytical solution for elliptic hole boundaries and for general anisotropic thermoelastic media is provided. A typical example of an elliptic hole boundary indented by a rigid stamp is solved in detail where the exact closed-form solutions are obtained and the correctness of the present results is checked analytically by their reduced forms such as those for the corresponding isothermal problems (Fan and Hwu, 1998) and those for the stress boundary value problems (Chao and Shen, 1998). The problem with a partially reinforced elliptic hole under a remote uniform heat flow is also solved completely and the correctness of the results is checked analytically as compared to the results for the stress boundary value problems (Chao and Shen, 1998).

2. Basic equations for plane thermoelasticity

With respect to a fixed rectangular coordinate system x_i, $i = 1, 2, 3$, let u_i, σ_{ij}, ε_{ij}, T, h_i be, respectively, the displacement, stress, strain, temperature, and heat flux. For the uncoupled steady-state thermoelastic problems, the heat conduction, energy equation, strain displacement relation, constitutive law, and the equations of equilibrium consist of (Nowacki, 1962)

$$h_i = -k_{ij}T_{,j} \tag{2.1}$$

$$h_{i,i} = -k_{ij}T_{,ij} = 0 \tag{2.2}$$

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \tag{2.3}$$

$$\sigma_{ij} = C_{ijks}u_{k,s} - \beta_{ij}T \tag{2.4}$$

$$\sigma_{ij,j} = C_{ijks}u_{k,sj} - \beta_{ij}T_{,j} = 0 \tag{2.5}$$

in which repeated indices imply summation, a comma stands for partial differentiation, and C_{ijks}, k_{ij}, β_{ij} are the elastic constants, heat conduction coefficients, and thermal moduli, respectively. For two-dimensional problems, the temperature T and the displacement u_i are independent of x_3. The general solution to Eq. (2.2), which is similar in form to that for antiplane deformation in anisotropic materials, can be expressed as

$$T = 2\,\mathrm{Re}[g'(z_\tau)], \quad z_\tau = x_1 + \tau x_2 \tag{2.6}$$

where g is an arbitrary function and the prime stands for differentiation with respect to its argument; Re denotes the real part of a complex function; the heat eigenvalue τ is the solution of

$$k_{22}\tau^2 + (k_{12} + k_{21})\tau + k_{11} = 0 \tag{2.7}$$

Here, the assumption of positive definiteness of the heat conduction coefficients has been adopted such that the heat eigenvalue τ cannot be a real number in Eq. (2.7).

Once the temperature function $g'(z_\tau)$ is determined, the general expressions for the displacement and stress functions can be expressed as (Ting, 1996)

$$u_\mathrm{i} = 2\,\mathrm{Re}\{A_{\mathrm{i}\alpha}f_\alpha(z_\alpha) + c_\mathrm{i}g(z_\tau)\} \tag{2.8}$$

$$\phi_\mathrm{i} = 2\,\mathrm{Re}\{B_{\mathrm{i}\alpha}f_\alpha(z_\alpha) + d_\mathrm{i}g(z_\tau)\} \tag{2.9}$$

$$z_\alpha = x_1 + p_\alpha x_2, \quad \alpha = 1,2,3 \tag{2.10}$$

$$\mathbf{A} = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}, \qquad \mathbf{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\} \tag{2.11}$$

Hereafter, bold symbols represent vectors or matrices. f is an arbitrary function, the elastic eigenvectors $\{\mathbf{a}, \mathbf{b}\}$ and the elastic eigenvalue p can be determined from the following eigenvalue problem

$$\mathbf{N}\xi = p\xi, \quad \mathbf{N} = \begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 \\ \mathbf{N}_3 & \mathbf{N}_1^\mathrm{T} \end{bmatrix}, \; \xi = \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} \tag{2.12}$$

where

$$\mathbf{N}_1 = -\mathbf{T}^{-1}\mathbf{R}^\mathrm{T}, \quad \mathbf{N}_2 = \mathbf{T}^{-1} = \mathbf{N}_2^\mathrm{T}, \quad \mathbf{N}_3 = \mathbf{R}\mathbf{T}^{-1}\mathbf{R}^\mathrm{T} - \mathbf{Q} = \mathbf{N}_3^\mathrm{T}$$
$$Q_{\mathrm{i}k} = C_{\mathrm{i}1k1}, \quad R_{\mathrm{i}k} = C_{\mathrm{i}1k2}, \quad T_{\mathrm{i}k} = C_{\mathrm{i}2k2},$$

the superscript T stands for the transpose of a matrix; while the vectors $\{\mathbf{c}, \mathbf{d}\}$ satisfy the following equation

$$\mathbf{N}\eta = \tau\eta - \gamma, \quad \gamma = \begin{bmatrix} \mathbf{0} & \mathbf{N}_2 \\ \mathbf{1} & \mathbf{N}_1^\mathrm{T} \end{bmatrix} \begin{Bmatrix} \beta_1 \\ \beta_2 \end{Bmatrix}, \; \eta = \begin{Bmatrix} \mathbf{c} \\ \mathbf{d} \end{Bmatrix} \tag{2.13}$$

where $\beta_1 = \{\beta_{\mathrm{i}1}\}$, $\beta_2 = \{\beta_{\mathrm{i}2}\}$ and $\mathbf{I}$ is the identity matrix. The vector ϕ given by Eq. (2.9) is the stress function which is related to the surface traction $\mathbf{t}$ by

$$\mathbf{t} = \frac{\partial \phi}{\partial s} \tag{2.14}$$

where s is the arc length measured along the curved boundary and the material is located on the right-hand side along the increasing path. Note that the general solutions in Eqs. (2.8) and (2.9) are for a state of plane strain and valid only when $p_1 \neq p_2 \neq p_3 \neq \tau$. However, for the special case in which the heat eigenvalue τ

becomes equal to a single or double elasticity eigenvalue, p_α, the solutions can still be derived by making use of the following identities

$$\mathbf{S} = \mathrm{i}(2\mathbf{A}\mathbf{B}^{\mathrm{T}} - \mathbf{I}), \quad \mathbf{H} = 2\mathrm{i}\mathbf{A}\mathbf{A}^{\mathrm{T}}, \quad \mathbf{L} = -2\mathrm{i}\mathbf{B}\mathbf{B}^{\mathrm{T}}, \quad \tilde{\gamma} = -\mathbf{L}\mathbf{c} + \mathbf{S}^{\mathrm{T}}\mathbf{d} - \mathrm{i}\mathbf{d}, \tag{2.15}$$

where the real matrices **S**, **H**, **L** and vector $\tilde{\gamma}$ can be determined from **c** and β_1, β_2 (Barnett and Lothe, 1973; Hwu, 1990).

3. Derivations of the general solution for curvilinear hole boundaries

Consider the thermoelastic problem of an anisotropic body containing an elliptic hole in which the displacement and the temperature are specified over a region L of the hole boundary while the remainder of the hole boundary L' is assumed to be traction-free and insulated from heat flow (see Fig. 1). These boundary conditions characterize the mixed boundary-value problems which would produce stress singularity at the tips of the region L as stated in the first section. In order to solve the problem with an awkwardly shaped region, a transformation $z_\alpha = m_\alpha(\zeta_\alpha)$ is introduced in the present problem which maps the points of a region S_ζ with a circular hole boundary in the ζ-plane onto the points of a region S_z with a curvilinear hole boundary in the z-plane. With this mapping function, all the solutions given in the last section are expressed in terms of the complex variable ζ instead of z. For the later use of derivation, a compact matrix form solution which satisfies all the basic equations given in Eqs. (2.1)–(2.5) is written as

$$T = 2\,\mathrm{Re}[g'(\zeta)] \tag{3.1}$$

$$Q = \int (h_1\,\mathrm{d}x_2 - h_2\,\mathrm{d}x_1) = 2\,\mathrm{Re}[\mathrm{i}kg'(\zeta)] \tag{3.2}$$

$$\mathbf{u} = 2\,\mathrm{Re}[\mathbf{A}\mathbf{f}(\zeta) + \mathbf{c}g(\zeta)] \tag{3.3}$$

$$\phi = 2\,\mathrm{Re}[\mathbf{B}\mathbf{f}(\zeta) + \mathbf{d}g(\zeta)] \tag{3.4}$$

with

$$\mathbf{f}(\zeta) = [f_1(\zeta), f_2(\zeta), f_3(\zeta)]^{\mathrm{T}} \tag{3.5}$$

where $k = (k_{11}k_{22} - k_{12}^2)^{1/2}$ and the argument has the generic form $\zeta = x_1 + p$ (or τ)x_2.

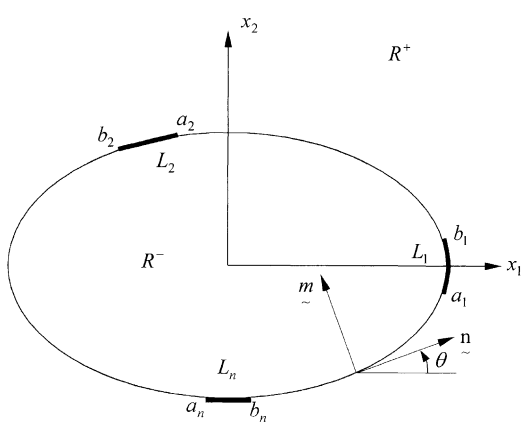

Fig. 1. An elliptic hole with mixed boundary conditions.

Once the solution of $\mathbf{f}(\zeta)$ (or $g(\zeta)$) is obtained for a boundary value problem, a replacement of $\zeta_1, \zeta_2, \zeta_3$ (or ζ_τ) should be made for each component function to calculate field quantities (Suo, 1990). On transforming to the region $|\zeta| \geqslant 1$ of the ζ-plane, the boundary conditions can then be expressed as

$$T_{,n}(\sigma) = 0, \qquad \mathbf{u}_{,n} = \hat{\mathbf{u}}'(\sigma) \quad \sigma \in L \tag{3.6}$$

$$h_m = \mathbf{t}_m = 0, \quad \sigma \notin L \tag{3.7}$$

where $\sigma = e^{i\psi}$ denotes the point on the unit circle of the ζ-plane; L is the union of n arcs $L_k = (a_k, b_k)$, $k = 1, 2, \dots, n$, $\hat{\mathbf{u}}'(\sigma)$ is the given function of the displacement gradient along the tangent direction $\mathbf{n}$; h_m and $\mathbf{t}_m$ are the heat flux and the traction function, respectively along the normal direction $\mathbf{m}$; $\mathbf{n}^T = (\cos\theta, \sin\theta, 0)$, $\mathbf{m}^T = (-\sin\theta, \cos\theta, 0)$; and θ is the angle measured counterclockwise between the tangent vector $\mathbf{n}$ and the positive x_1-axis (Fig. 1).

3.1. The thermal field

We first consider the thermal problem with the unknown function $g(\zeta)$ which satisfies the boundary conditions (3.6) and (3.7). To solve this function the temperature gradient h_m is expressed in terms of $g''(\zeta)$ as

$$\begin{aligned} h_m &= \frac{\partial Q}{\partial n} = \frac{\partial}{\partial n}\left[ikg'(\zeta) - i\overline{kg'(\zeta)}\right] \\ &= \frac{d}{d\zeta}\left[ikg'(\zeta)\right]\frac{\partial \zeta}{\partial \psi}\frac{\partial \psi}{\partial z_\tau}\left[\frac{\partial z_\tau}{\partial x_1}\frac{\partial x_1}{\partial n} + \frac{\partial z_\tau}{\partial x_2}\frac{\partial x_2}{\partial n}\right] - \frac{d}{d\bar{\zeta}}\left[i\overline{kg'(\zeta)}\right]\frac{\partial \bar{\zeta}}{\partial \psi}\frac{\partial \psi}{\partial \bar{z}_\tau}\left[\frac{\partial \bar{z}_\tau}{\partial x_1}\frac{\partial x_1}{\partial n} + \frac{\partial \bar{z}_\tau}{\partial x_2}\frac{\partial x_2}{\partial n}\right], \quad \zeta \to \sigma^+ \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} &\frac{\partial x_1}{\partial n} = \cos\theta, \quad \frac{\partial x_2}{\partial n} = \sin\theta, \quad \frac{\partial \zeta}{\partial \psi} = i\zeta, \quad \frac{\partial z_\tau}{\partial x_1} = 1, \quad \frac{\partial z_\tau}{\partial x_2} = \tau, \quad \frac{\partial x_1}{\rho \partial \psi} = -\cos\theta \\ &\frac{\partial x_2}{\rho \partial \psi} = -\sin\theta, \qquad \frac{\partial \psi}{\partial z_\tau} = \frac{-1}{\rho(\cos\theta + \tau\sin\theta)}, \quad \rho = \sqrt{\left|\frac{\partial x_1}{\partial \psi}\right|^2 + \left|\frac{\partial x_2}{\partial \psi}\right|^2} \end{aligned} \tag{3.9}$$

On substituting Eq. (3.9) into Eq. (3.8) we find

$$h_m = \frac{k\zeta}{\rho}g''(\zeta) + \frac{k}{\rho\zeta}\overline{g''(\zeta)}, \quad \zeta \to \sigma^+ \tag{3.10}$$

or

$$h_m = \frac{k\sigma}{\rho}g''(\sigma^+) + \frac{k}{\rho\sigma}\overline{g''\left(\frac{1}{\bar{\sigma}^-}\right)}, \tag{3.11}$$

where the superscript + (or −) denotes the boundary point is approached from the region S^+ (or S^-). Using the properties of holomorphic functions and applying the method of analytical continuation (England, 1971), we may introduce $\theta_0(\zeta)$ in the form

$$\theta_0(\zeta) = \begin{cases} \frac{k\zeta}{\rho}g''(\zeta), & \zeta \in S^+ \\ \frac{-k}{\rho\zeta}\overline{g''\left(\frac{1}{\bar{\zeta}}\right)}, & \zeta \in S^- \end{cases} \tag{3.12}$$

With the definition of Eq. (3.12), Eq. (3.11) can be rewritten as

$$h_m = \theta_0(\sigma^+) - \theta_0(\sigma^-) \tag{3.13}$$

In a similar way, the temperature gradient in the direction of **n** can be obtained as

$$\frac{\mathrm{d}T}{\mathrm{d}n} = \frac{\mathrm{i}}{k}[\theta_0(\sigma^+) + \theta_0(\sigma^-)] \tag{3.14}$$

Substituting the boundary conditions (3.6) and (3.7) into Eqs. (3.13) and (3.14), we obtain the following Hilbert problem

$$\theta_0(\sigma^+) + \theta_0(\sigma^-) = 0, \quad \sigma \in L \tag{3.15}$$

$$\theta_0(\sigma^+) - \theta_0(\sigma^-) = 0, \quad \sigma \notin L \tag{3.16}$$

The solution can be easily found as

$$\theta_0(\zeta) = X_0(\zeta)p_0(\zeta) \tag{3.17}$$

where $p_0(\zeta)$ is a polynomial function and $X_0(\zeta)$ is the basic Plemelj function defined as

$$X_0(\zeta) = \prod_{j=1}^{n}(\zeta - a_j)^{-1/2}(\zeta - b_j)^{-1/2} \tag{3.18}$$

The determination of $p_0(\zeta)$ may be found from the behavior of the temperature function at infinity and origin which has the form for large $|z|$ as (Chao and Shen, 1998)

$$g(z) = \frac{h_0}{2}z_\tau^2 + r_0 z_\tau \log z_\tau + l_0 \log z_\tau + \mathrm{O}(1) \tag{3.19}$$

where h_0 is related to the uniform heat flux applied at infinity defined as (Chao and Shen, 1998)

$$h_0 = \frac{q_0(\cos\gamma_0 + \bar{\tau}\sin\gamma_0)}{\mathrm{i}k(\tau - \bar{\tau})} \tag{3.20}$$

with q_0 being the magnitude of heat flux and γ_0 the angle measured from the direction of heat flux with respect to the positive x_1-axis; the constant r_0 in Eq. (3.19) can be found from the condition of the resultant heat flow $\hat{Q}$ applied at the entire body as well as the condition that the temperature field is required to be single-valued.

These conditions are expressed as

$$[Q(\zeta)]_c = 2\,\mathrm{Re}[\mathrm{i}kg'(\zeta)]_c = \hat{Q}, \tag{3.21}$$

$$[T(\zeta)]_c = 2\,\mathrm{Re}[g'(\zeta)]_c = 0 \tag{3.22}$$

where $[g(\zeta)]_c$ denotes the jump of a complex function $g(\zeta)$ when enclosing any contour c.

Substituting Eq. (3.19) into Eqs. (3.21) and (3.22) and solving for r_0, we have

$$r_0 = -\frac{\hat{Q}}{4\pi k} \tag{3.23}$$

The remaining unknown constant l_0 in Eq. (3.19) will be determined once the geometry of a curvilinear hole is given.

With the transformation $z_\tau = m_\tau(\zeta)$ and the definition of Eq. (3.12), the infinity and origin conditions for $\theta_0(\zeta)$ can be found by using Eq. (3.19) as

$$\theta_0(\zeta) = \frac{k\zeta}{\rho}g''(\zeta) = \frac{k\zeta}{\rho}\left[h_0 + \frac{r_0}{m_\tau(\zeta)} - \frac{l_0}{(m_\tau(\zeta))^2}\right]m_\tau'(\zeta) + \mathrm{O}\left(\frac{1}{\zeta}\right), \quad |\zeta| \to \infty \tag{3.24}$$

and

$$\theta_0(\zeta) = \frac{-k}{\rho\zeta}\overline{g''\left(\frac{1}{\bar{\zeta}}\right)} = \frac{-k}{\rho\zeta}\left[\overline{h_0} + \frac{\overline{r_0}}{\overline{m_\tau}(1/\zeta)} - \frac{\overline{l_0}}{(\overline{m_\tau}(1/\zeta))^2}\right]\overline{m'_\tau}\left(\frac{1}{\zeta}\right) + \mathrm{O}(\zeta), \quad |\zeta| \to 0 \tag{3.25}$$

3.2. The stress field

Having the temperature function, we now intend to find the stress function $\mathbf{f}(\zeta)$ in Eqs. (3.3) and (3.4) which satisfies the boundary conditions (3.6) and (3.7). In a way similar to the previous approach, the surface traction $\mathbf{t}_m$ along the hole boundary can be expressed as

$$\begin{aligned}\mathbf{t}_m &= \frac{\partial \boldsymbol{\phi}}{\partial n} = \frac{\partial}{\partial n}\left[\mathbf{B}\mathbf{f}(\zeta) + \overline{\mathbf{B}}\overline{\mathbf{f}(\zeta)} + \mathbf{d}g(\zeta) + \overline{\mathbf{d}}\overline{g(\zeta)}\right] \\ &= \frac{-\mathrm{i}\zeta}{\rho}[\mathbf{B}\mathbf{f}'(\zeta) + \mathbf{d}g'(\zeta)] + \frac{\mathrm{i}}{\zeta\rho}\left[\overline{\mathbf{B}}\overline{\mathbf{f}'(\zeta)} + \overline{\mathbf{d}}\overline{g'(\zeta)}\right], \quad \zeta \to \sigma^+\end{aligned} \tag{3.26}$$

where we have used the relations (3.9) by the replacement of z_τ and τ with z_α and p_α, respectively. Eq. (3.26) can then be rewritten as

$$\mathbf{t}_m = \frac{-\mathrm{i}\sigma}{\rho}\left[\mathbf{B}\mathbf{f}'(\sigma^+) + \mathbf{d}g'(\sigma^+)\right] + \frac{\mathrm{i}}{\rho\sigma}\left[\overline{\mathbf{B}}\overline{\mathbf{f}'\left(\frac{1}{\overline{\sigma^-}}\right)} + \overline{\mathbf{d}}\overline{g'\left(\frac{1}{\overline{\sigma^-}}\right)}\right] \tag{3.27}$$

Now we introduce a new holomorphic function as

$$\boldsymbol{\Theta}'(\zeta) = \begin{cases}\zeta[\mathbf{B}\mathbf{f}'(\zeta) + \mathbf{d}g'(\zeta)], & \zeta \in S^+ \\ \frac{1}{\zeta}\left[\overline{\mathbf{B}}\overline{\mathbf{f}'(1/\bar{\zeta})} + \overline{\mathbf{d}}\overline{g'(1/\bar{\zeta})}\right], & \zeta \in S^-\end{cases} \tag{3.28}$$

With the definition of Eq. (3.28), Eq. (3.26) can be replaced by

$$\boldsymbol{\Theta}'(\sigma^+) - \boldsymbol{\Theta}'(\sigma^-) = \mathrm{i}\rho\mathbf{t}_m \tag{3.29}$$

Similarly, the displacement gradients along the tangent direction $\mathbf{n}$ of the hole boundary can be obtained as

$$\boldsymbol{\Theta}'(\sigma^+) + \mathbf{M}\overline{\mathbf{M}}^{-1}\boldsymbol{\Theta}'(\sigma^-) - 2\mathbf{M}\,\mathrm{Im}\left[(\mathbf{c} - \mathbf{A}\mathbf{B}^{-1}\mathbf{d})\sigma g'(\sigma^+)\right] = -\rho\mathbf{M}\mathbf{u}_{,n} \tag{3.30}$$

where $\mathbf{M} = -\mathbf{i}\mathbf{B}\mathbf{A}^{-1}$ is the impedance matrix; Im denotes the imaginary part of a complex function. Using the boundary conditions (3.6) and (3.7), we obtain the following Hilbert problem from Eqs. (3.29) and (3.30) as

$$\boldsymbol{\Theta}'(\sigma^+) + \mathbf{M}\overline{\mathbf{M}}^{-1}\boldsymbol{\Theta}'(\sigma^-) = \mathbf{M}\left\{-\rho\hat{\mathbf{u}}' + 2\,\mathrm{Im}\left[(\mathbf{c} - \mathbf{A}\mathbf{B}^{-1}\mathbf{d})\sigma g'(\sigma^+)\right]\right\}, \quad \sigma \in L \tag{3.31}$$

$$\boldsymbol{\Theta}'(\sigma^+) - \boldsymbol{\Theta}'(\sigma^-) = 0, \quad \sigma \notin L \tag{3.32}$$

The general solution to this Hilbert problem is

$$\boldsymbol{\Theta}'(\zeta) = \frac{1}{2\pi\mathrm{i}}\mathbf{X}(\zeta)\int_L \frac{1}{t-\zeta}[\mathbf{X}^+(t)]^{-1}\mathbf{M}\left\{-\rho\hat{\mathbf{u}}' + 2\,\mathrm{Im}\left[(\mathbf{c} - \mathbf{A}\mathbf{B}^{-1}\mathbf{d})tg'(t)\right]\right\}\mathrm{d}t + \mathbf{X}(\zeta)\mathbf{p}(\zeta) \tag{3.33}$$

where $\mathbf{p}(\zeta)$ is a polynomial with an appropriate singularity at infinity and origin; $\mathbf{X}(\zeta)$ is the basic Plemelj function defined as

$$\mathbf{X}(\zeta) = \boldsymbol{\Lambda}\boldsymbol{\Gamma}(\zeta) \tag{3.34}$$

where

$$\mathbf{\Lambda} = [\lambda_1, \lambda_2, \lambda_3], \quad \mathbf{\Gamma}(\zeta) = \left\langle\left\langle \prod_{j=1}^{n} (\zeta - a_j)^{-(1+\delta_\alpha)} (\zeta - b_j)^{\delta_\alpha} \right\rangle\right\rangle \tag{3.35}$$

The angular brackets $\langle\langle\ \rangle\rangle$ stands for the diagonal matrix in which each component is varied according to the Greek index α; δ_α and λ_α, $\alpha = 1, 2, 3$ are the eigenvalues and eigenvectors of (Fan and Hwu, 1996)

$$\left(\overline{\mathbf{M}}^{-1} + e^{2\pi i \delta}\mathbf{M}^{-1}\right)\lambda = 0 \tag{3.36}$$

In order to find the arbitrary polynomial vector $\mathbf{p}(\zeta)$, we consider the complex function vector $\mathbf{f}(z)$ which has the form for large $|\zeta|$ as

$$\mathbf{f}(z) = \langle\langle z_\alpha \rangle\rangle \mathbf{q} + \langle\langle \log z_\alpha \rangle\rangle \mathbf{r} + \langle\langle H(z_\alpha) \rangle\rangle + O(1) \tag{3.37}$$

where $\mathbf{q}$ is related to the stresses σ_{ij}^{∞} and strains $\varepsilon_{ij}^{\infty}$ at infinity as (Eshelby et al., 1953; Ting, 1988)

$$\mathbf{q} = \mathbf{A}^{\mathrm{T}}\mathbf{t}_2^{\infty} + \mathbf{B}^{\mathrm{T}}\varepsilon_1^{\infty} \tag{3.38}$$

with

$$\mathbf{t}_2^{\infty} = \begin{Bmatrix} \sigma_{12}^{\infty} \\ \sigma_{22}^{\infty} \\ \sigma_{32}^{\infty} \end{Bmatrix}, \qquad \varepsilon_1^{\infty} = \begin{Bmatrix} \varepsilon_{11}^{\infty} \\ \varepsilon_{12}^{\infty} \\ 2\varepsilon_{13}^{\infty} \end{Bmatrix} \tag{3.39}$$

The complex constant $\mathbf{r}$ is related to the resultant force $\hat{\mathbf{p}}$ applied on the entire body and the complex function $H(z_\alpha)$ will be determined to counterbalance the singular term $r_0 z_\tau \log z_\tau$ of the temperature function given in Eq. (3.19). For convenience of the calculation, the complex function $H(z_\alpha)$ is chosen as the form $\mathbf{h}(\zeta)\log\zeta$ and using the transformation $z_\alpha = m_\alpha(\zeta)$, the stress function $\mathbf{f}(\zeta)$ in Eq. (3.37) becomes

$$\mathbf{f}(\zeta) = \langle\langle m_\alpha(\zeta) \rangle\rangle \mathbf{q} + \langle\langle \log(m_\alpha(\zeta)) \rangle\rangle \mathbf{r} + \langle\langle \log\zeta \rangle\rangle \mathbf{h}(\zeta) + O(1) \tag{3.40}$$

In order to determine the unknown constant $\mathbf{r}$ and unknown function $\mathbf{h}(\zeta)$ in Eq. (3.40), the condition of single-valued displacements and the condition of the resultant force $\hat{\mathbf{p}}$ applied over the entire body must be satisfied to yield

$$\left[\mathbf{A}\mathbf{f}(\zeta) + \overline{\mathbf{A}}\,\overline{\mathbf{f}(\zeta)} + \mathbf{c}g(\zeta) + \overline{\mathbf{c}}\,\overline{g(\zeta)}\right]_{\mathrm{c}} = 0 \tag{3.41}$$

$$\left[\mathbf{B}\mathbf{f}(\zeta) + \overline{\mathbf{B}}\,\overline{\mathbf{f}(\zeta)} + \mathbf{d}g(\zeta) + \overline{\mathbf{d}}\,\overline{g(\zeta)}\right]_{\mathrm{c}} = \hat{\mathbf{p}} \tag{3.42}$$

By direct substitution of Eqs. (3.19) and (3.40) into Eqs. (3.41) and (3.42), we may immediately have the following equations

$$\begin{aligned} &\mathbf{A}\mathbf{r} - \overline{\mathbf{A}}\overline{\mathbf{r}} + \mathbf{c}l_0 - \overline{\mathbf{c}}\overline{l_0} = 0 \\ &\mathbf{B}\mathbf{r} - \overline{\mathbf{B}}\overline{\mathbf{r}} + \mathbf{d}l_0 - \overline{\mathbf{d}}\,\overline{l_0} = \frac{\hat{\mathbf{p}}}{2\pi i} \\ &\mathbf{A}\mathbf{h}(\zeta) - \overline{\mathbf{A}}\,\overline{\mathbf{h}(\zeta)} + \mathbf{c}r_0 m_\tau(\zeta) - \overline{\mathbf{c}}\,\overline{r_0}\,\overline{m_\tau(\zeta)} = 0 \\ &\mathbf{B}\mathbf{h}(\zeta) - \overline{\mathbf{B}}\,\overline{\mathbf{h}(\zeta)} + \mathbf{d}r_0 m_\tau(\zeta) - \overline{\mathbf{d}}\,\overline{r_0}\,\overline{m_\tau(\zeta)} = 0 \end{aligned} \tag{3.43}$$

Solving for $\mathbf{r}$ and $\mathbf{h}(\zeta)$ in Eq. (3.43), we obtain

$$\mathbf{r} = \frac{1}{2\pi i}\mathbf{A}^{\mathrm{T}}\hat{\mathbf{p}} - \mathbf{A}^{\mathrm{T}}\left(\mathbf{d}l_0 - \overline{\mathbf{d}}\,\overline{l_0}\right) - \mathbf{B}^{\mathrm{T}}\left(\mathbf{c}l_0 - \overline{\mathbf{c}}\overline{l_0}\right) \tag{3.44}$$

and

$$\mathbf{h}(\zeta) = -\left(\mathbf{B}^{\mathrm{T}}\mathbf{c} + \mathbf{A}^{\mathrm{T}}\mathbf{d}\right) r_0 m_\tau(\zeta) + \left(\mathbf{B}^{\mathrm{T}}\bar{\mathbf{c}} + \mathbf{A}^{\mathrm{T}}\bar{\mathbf{d}}\right)\overline{r_0}\,\overline{m_\tau(\zeta)} \tag{3.45}$$

With the definition of Eq. (3.28), the infinity and origin conditions for $\mathbf{\Theta}'(\zeta)$ can be obtained by using Eqs. (3.19) and (3.40) as

$$\mathbf{\Theta}'(\zeta) = \mathbf{B}\left\{\langle\langle \zeta m'_\alpha(\zeta)\rangle\rangle \mathbf{q} + \left\langle\left\langle \frac{\zeta m'_\alpha(\zeta)}{m_\alpha(\zeta)}\right\rangle\right\rangle \mathbf{r} + \langle\langle \zeta \log \zeta\rangle\rangle \mathbf{h}'(\zeta) + \mathbf{h}(\zeta)\right\} + \mathbf{d}\left[h_0 m_\tau(\zeta) + r_0 \log\left(m_\tau(\zeta)\right) + r_0 + \frac{l_0}{m_\tau(\zeta)}\right]\zeta m'_\tau(\zeta) + \mathrm{O}\left(\frac{1}{\zeta}\right), \quad |\zeta| \to \infty \tag{3.46}$$

and

$$\mathbf{\Theta}'(\zeta) = \overline{\mathbf{B}}\left\{\left\langle\left\langle \frac{1}{\zeta}\overline{m'_\alpha}\left(\frac{1}{\zeta}\right)\right\rangle\right\rangle \bar{\mathbf{q}} + \left\langle\left\langle \frac{\overline{m'_\alpha}\left(\frac{1}{\zeta}\right)}{\zeta \overline{m_\alpha}(1/\zeta)}\right\rangle\right\rangle \bar{\mathbf{r}} - \left\langle\left\langle \frac{\log \zeta}{\zeta}\right\rangle\right\rangle \overline{\mathbf{h}'}\left(\frac{1}{\zeta}\right) + \bar{\mathbf{h}}\left(\frac{1}{\zeta}\right)\right\} + \bar{\mathbf{d}}\left[\overline{h_0}\,\overline{m_\tau}\left(\frac{1}{\zeta}\right) + \overline{r_0}\log\left(\overline{m_\tau}\left(\frac{1}{\zeta}\right)\right) + \overline{r_0} + \frac{\overline{l_0}}{\overline{m_\tau}(1/\zeta)}\right]\frac{1}{\zeta}\overline{m'_\tau}\left(\frac{1}{\zeta}\right) + \mathrm{O}(\zeta), \quad |\zeta| \to 0 \tag{3.47}$$

Thus far we have completed the derivations of the general solution; the only unknown functions $p_0(\zeta)$ in Eq. (3.17) and $\mathbf{p}(\zeta)$ in Eq. (3.33) will be obtained with the help of Eqs. (3.24)–(3.47).

4. Rigid stamp indentation on an elliptic hole boundary

Although the solutions presented in the last section are derived for any curvilinear boundary, the exact closed-form solutions can only be obtained when the transformation functions are single-valued. A typical geometry of curvilinear boundaries is the one with elliptic boundaries whose transformation function

$$z_\tau = m_\tau(\zeta_\tau) = \tfrac{1}{2}\left\{(a - \mathrm{i}b\tau)\zeta_\tau + (a + \mathrm{i}b\tau)\zeta_\tau^{-1}\right\} \tag{4.1}$$

or

$$z_\alpha = m_\alpha(\zeta_\alpha) = \tfrac{1}{2}\left\{(a - \mathrm{i}bp_\alpha)\zeta_\alpha + (a + \mathrm{i}bp_\alpha)\zeta_\alpha^{-1}\right\} \tag{4.2}$$

is found to be single-valued when the points outside the elliptic hole with the major length $2a$ and the minor length $2b$ of the z-plane are designated to map onto the points outside the unit circle of the ζ-plane. Suppose that the hole is subjected to a resultant heat flow $\hat{Q}$ approached from the negative x_1-axis and is loaded by a rigid stamp with a resultant force $\hat{\mathbf{p}}$ along the segment between $(a\cos\varphi, -b\sin\varphi)$ and $(a\cos\varphi, b\sin\varphi)$ which is mapped onto an arc $L = (\mathrm{e}^{-\mathrm{i}\varphi}, \mathrm{e}^{\mathrm{i}\varphi})$ in the ζ-plane. With this specification, the general solution given in Eq. (3.17) becomes

$$\theta_0(\zeta) = X_0(\zeta)p_0(\zeta) \tag{4.3}$$

where

$$X_0(\zeta) = \left(\zeta - \mathrm{e}^{-\mathrm{i}\varphi}\right)^{-1/2}\left(\zeta - \mathrm{e}^{\mathrm{i}\varphi}\right)^{-1/2} \tag{4.4}$$

and the polynomial $p_0(\zeta)$ is determined from the infinity and origin conditions of the temperature function. By direct substitution of Eq. (4.1) into Eqs. (3.24) and (3.25) and knowing that $h_0 = 0$ in the present problem, we have

$$\theta_0(\zeta) = \frac{k}{\rho}r_0 + \mathrm{O}\left(\frac{1}{\zeta}\right), \quad |\zeta| \to \infty \tag{4.5}$$

and

$$\theta_0(\zeta) = -\frac{k}{\rho} r_0 + \mathrm{O}(\zeta), \quad |\zeta| \to 0 \tag{4.6}$$

where r_0 is given by Eq. (3.23).

Moreover, $X_0(\zeta)$ given in Eq. (4.4) has the form for large $|\zeta|$ as

$$X_0(\zeta) = \frac{1}{\zeta} + \frac{\cos\varphi}{\zeta^2} + \mathrm{O}\left(\frac{1}{\zeta^3}\right) \tag{4.7}$$

and for small $|\zeta|$ as

$$X_0(\zeta) = -1 - \zeta\cos\varphi + \mathrm{O}(\zeta^2) \tag{4.8}$$

Using the properties given in Eqs. (4.5)–(4.8), Eq. (4.3) yields to find the polynomial $p_0(\zeta)$ as

$$p_0(\zeta) = \frac{kr_0}{\rho}(1+\zeta) \tag{4.9}$$

Substituting Eq. (4.9) into Eq. (4.3) and applying Eq. (3.12), we find

$$\theta_0(\zeta) = \frac{k\zeta}{\rho} g''(\zeta) = \frac{kr_0(1+\zeta)}{\rho\sqrt{(\zeta - \mathrm{e}^{-\mathrm{i}\varphi})(\zeta - \mathrm{e}^{\mathrm{i}\varphi})}} \tag{4.10}$$

and the temperature function $g'(\zeta)$ can be determined by integrating Eq. (4.10) with respect to ζ as

$$g'(\zeta) = r_0\left\{\sinh^{-1}\left(\frac{\zeta - \cos\varphi}{\sin\varphi}\right) - \sinh^{-1}\left(\frac{1-\cos\varphi}{\zeta\sin\varphi}\right)\right\} \tag{4.11}$$

In the present problem we assume $\hat{\mathbf{u}}'(\sigma) = 0$ which implies the profile of the stamp is compatible with that of the hole boundary. With this condition and the obtained temperature function (4.11), the general solution given in Eq. (3.33) becomes

$$\begin{aligned}\mathbf{\Theta}'(\zeta) = \frac{r_0}{\pi\mathrm{i}}\mathbf{X}(\zeta)\int_L \frac{1}{t-\zeta}[\mathbf{X}^+(t)]^{-1}\mathbf{M}\,\mathrm{Im}\bigg\{(\mathbf{c} - \mathbf{A}\mathbf{B}^{-1}\mathbf{d})t\left[\sinh^{-1}\left(\frac{t-\cos\varphi}{\sin\varphi}\right)\right. \\ \left. - \sinh^{-1}\left(\frac{1-\cos\varphi}{t\sin\varphi}\right)\right]\left(a_{1\tau} - \frac{a_{2\tau}}{t^2}\right)\bigg\}\,\mathrm{d}t + \mathbf{X}(\zeta)\mathbf{p}(\zeta)\end{aligned} \tag{4.12}$$

where $a_{1\tau} = (a - \mathrm{i}b\tau)/2$, $a_{2\tau} = (a + \mathrm{i}b\tau)/2$, and $\mathbf{X}(\zeta)$ is defined as

$$\mathbf{X}(\zeta) = \mathbf{\Lambda}\mathbf{\Gamma}(\zeta) = \mathbf{\Lambda}\left\langle\left\langle(\zeta - \mathrm{e}^{-\mathrm{i}\varphi})^{-(1/2)-\mathrm{i}\varepsilon_\alpha}(\zeta - \mathrm{e}^{\mathrm{i}\varphi})^{-(1/2)+\mathrm{i}\varepsilon_\alpha}\right\rangle\right\rangle, \quad \mathrm{i}\varepsilon_\alpha = \tfrac{1}{2} + \delta_\alpha \tag{4.13}$$

To obtain the polynomial vector $\mathbf{p}(\zeta)$, the properties of the stress function $\mathbf{\Theta}'(\zeta)$ at infinity and origin must be used. By direct substitution of Eq. (4.1) into Eqs. (3.46) and (3.47), we find

$$\mathbf{\Theta}'(\zeta) = \mathbf{B}\left\{\mathbf{r} + \frac{\zeta}{2}\langle\langle a - \mathrm{i}bp_\alpha\rangle\rangle\mathbf{q} + \mathrm{O}\left(\frac{1}{\zeta}\right)\right\}, \quad |\zeta| \to \infty \tag{4.14}$$

and

$$\mathbf{\Theta}'(\zeta) = \overline{\mathbf{B}}\left\{\overline{\mathbf{r}} + \frac{1}{2\zeta}\langle\langle a + \mathrm{i}b\overline{p}_\alpha\rangle\rangle\overline{\mathbf{q}} + \mathrm{O}(\zeta)\right\}, \quad |\zeta| \to 0 \tag{4.15}$$

Notice that the unwelcomed singular term $\zeta\log\zeta$, which has been cancelled out by the proper choice of the function $h(\zeta)$ defined in Eq. (3.45), does not appear in Eqs. (4.14) and (4.15).

In addition, $\boldsymbol{\Gamma}(\zeta)$ given in Eq. (4.13) can be expanded for large $|\zeta|$ as

$$\boldsymbol{\Gamma}(\zeta) = \frac{1}{\zeta} + \frac{\mathbf{G}_1}{\zeta^2} + \frac{\mathbf{G}_2}{\zeta^3} + \frac{\mathbf{G}_3}{\zeta^4} + \mathrm{O}\left(\frac{1}{\zeta^5}\right) \tag{4.16}$$

and for small $|\zeta|$ as

$$\boldsymbol{\Gamma}(\zeta) = \langle\langle -\mathrm{e}^{-2\varphi\varepsilon_\alpha}\rangle\rangle\{1 + \mathbf{G}_4\zeta + \mathbf{G}_5\zeta^2 + \mathbf{G}_6\zeta^3\} + \mathrm{O}(\zeta^4) \tag{4.17}$$

where

$$\begin{aligned}
\mathbf{G}_1 &= \langle\langle \cos\varphi + 2\varepsilon_\alpha \sin\varphi\rangle\rangle,\\
\mathbf{G}_2 &= \langle\langle \tfrac{1}{4}(1 + 4\varepsilon_\alpha^2)(1 - \cos 2\varphi) + \cos 2\varphi + 2\varepsilon_\alpha \sin 2\varphi\rangle\rangle,\\
\mathbf{G}_3 &= \left\langle\left\langle \frac{1}{3}\left[\left(-\frac{23}{4} + \varepsilon_\alpha^2\right)\varepsilon_\alpha \sin 3\varphi + \left(\frac{15}{8} - \frac{9}{2}\varepsilon_\alpha^2\right)\cos 3\varphi\right] + \left(\frac{3}{8} + \frac{3}{2}\varepsilon_\alpha^2\right)\cos\varphi - \left(\frac{1}{4} + \varepsilon_\alpha^2\right)\phi_\alpha \sin\varphi\right\rangle\right\rangle,\\
\mathbf{G}_4 &= \langle\langle \cos\varphi - 2\varepsilon_\alpha \sin\varphi\rangle\rangle,\\
\mathbf{G}_5 &= \langle\langle \tfrac{1}{4}(1 + 4\varepsilon_\alpha^2)(1 - \cos 2\varphi) + \cos 2\varphi - 2\varepsilon_\alpha \sin 2\varphi\rangle\rangle,\\
\mathbf{G}_6 &= \left\langle\left\langle \frac{1}{3}\left[\left(\frac{23}{4} - \varepsilon_\alpha^2\right)\varepsilon_\alpha \sin 3\varphi + \left(\frac{15}{8} - \frac{9}{2}\varepsilon_\alpha^2\right)\cos 3\varphi\right] + \left(\frac{3}{8} + \frac{3}{2}\varepsilon_\alpha^2\right)\cos\varphi + \left(\frac{1}{4} + \varepsilon_\alpha^2\right)\varepsilon_\alpha \sin\varphi\right\rangle\right\rangle
\end{aligned} \tag{4.18}$$

Using the properties given in Eqs. (4.14)–(4.17), Eq. (4.12) yields to determine the polynomial vector $\mathbf{p}(\zeta)$ as

$$\mathbf{p}(\zeta) = \mathbf{e}_2\zeta^2 + \mathbf{e}_1\zeta + \mathbf{e}_0 + \mathbf{e}_{-1}\zeta^{-1} \tag{4.19}$$

where the constant coefficients $\mathbf{e}_{-1}, \mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2$ satisfy the following relation (see Appendix A)

$$\begin{aligned}
&2\boldsymbol{\Lambda}\mathbf{e}_2 = \mathbf{B}\langle\langle a - \mathrm{i}bp_\alpha\rangle\rangle\mathbf{q},\\
&\boldsymbol{\Lambda}\mathbf{e}_1 + \boldsymbol{\Lambda}\langle\langle(\tfrac{1}{2} - \mathrm{i}\varepsilon_\alpha)\mathrm{e}^{\mathrm{i}\varphi} + (\tfrac{1}{2} + \mathrm{i}\varepsilon_\alpha)\mathrm{e}^{-\mathrm{i}\varphi}\rangle\rangle\mathbf{e}_2 = \mathbf{Br}, \quad \boldsymbol{\Lambda}\langle\langle \mathrm{e}^{-2\varphi\varepsilon_\alpha}\rangle\rangle\mathbf{e}_{-1} = -\tfrac{1}{2}\overline{\mathbf{B}}\langle\langle a + \mathrm{i}b\overline{p_\alpha}\rangle\rangle\overline{\mathbf{q}},\\
&\boldsymbol{\Lambda}\{\langle\langle \mathrm{e}^{-2\varphi\varepsilon_\alpha}\rangle\rangle\mathbf{e}_0 + \langle\langle \mathrm{e}^{-2\varphi\varepsilon_\alpha}[(\tfrac{1}{2} + \mathrm{i}\varepsilon_\alpha)\mathrm{e}^{\mathrm{i}\varphi} + (\tfrac{1}{2} - \mathrm{i}\varepsilon_\alpha)\mathrm{e}^{-\mathrm{i}\varphi}]\rangle\rangle\mathbf{e}_{-1}\} = -\overline{\mathbf{B}}\overline{\mathbf{r}}
\end{aligned} \tag{4.20}$$

Solving Eq. (4.20) for the four unknowns $\mathbf{e}_{-1}, \mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2$, the problem is completely solved.

Since the solutions obtained in this paper are new and no other analytical solution is available in the literature, the correctness of the present results can only be checked analytically by their reduced forms such as those for the corresponding isothermal problems and those for stress boundary value problems. We first consider a special case that an anisotropic body containing an elliptic hole is subjected to a resultant force $\hat{\mathbf{p}}$ applied on the rigid stamp. In this case $\mathbf{q} = 0$ and the unknown constant coefficients given in Eq. (4.20) become

$$\mathbf{e}_{-1} = 0, \quad \mathbf{e}_0 = -\langle\langle \mathrm{e}^{2\varphi\varepsilon_\alpha}\rangle\rangle\boldsymbol{\Lambda}^{-1}\overline{\mathbf{B}}\overline{\mathbf{r}}, \quad \mathbf{e}_1 = \boldsymbol{\Lambda}^{-1}\mathbf{Br}, \quad \mathbf{e}_2 = 0 \tag{4.21}$$

where $\mathbf{r} = (1/2\pi\mathrm{i})\mathbf{A}^{\mathrm{T}}\hat{\mathbf{p}}$ as found from Eq. (3.44).

By combining Eqs. (3.28), (4.12), (4.19) and (4.21), the final expression of $\mathbf{f}'(\zeta)$ is obtained as

$$\mathbf{f}'(\zeta) = \frac{1}{2\pi\mathrm{i}}\mathbf{B}^{-1}\boldsymbol{\Lambda}\boldsymbol{\Gamma}(\zeta)\left[\boldsymbol{\Lambda}^{-1}\mathbf{B}\mathbf{A}^{\mathrm{T}} + \left\langle\left\langle \frac{\mathrm{e}^{2\varphi\varepsilon_\varepsilon}}{\zeta}\right\rangle\right\rangle\boldsymbol{\Lambda}^{-1}\overline{\mathbf{B}}\,\overline{\mathbf{A}}^{\mathrm{T}}\right]\hat{\mathbf{p}} \tag{4.22}$$

which is identical to that provided by Fan and Hwu (1998).

Next we consider an anisotropic body containing an elliptic hole which is subjected to a remote uniform heat flow with an angle γ_0 measured from the positive x_1-axis. If the whole boundary of the elliptic hole is traction-free and insulated, the displacement-prescribed and temperature-prescribed segment L will vanish, and the present mixed boundary-value problem reduces to a stress boundary-value problem. In such a problem, the Plemelj functions $X_0(\zeta)$ in Eq. (3.18) becomes a unity and the solution will be reduced to a polynomial form

$$\theta_0(\zeta) = p_0(\zeta) \tag{4.23}$$

where $p_0(\zeta)$ can be determined from the infinity and origin conditions of the temperature function. Substituting Eq. (4.1) into Eqs. (3.24) and (3.25) with knowing that $r_0 = 0$, we have

$$\theta_0(\zeta) = \frac{k\zeta}{\rho}\left[h_0 - \frac{l_0}{(m_\tau(\zeta))^2}\right] m'_\tau(\zeta) + \mathrm{O}\left(\frac{1}{\zeta}\right) = \frac{k}{\rho} a_{1\tau} h_0 \zeta + \mathrm{O}\left(\frac{1}{\zeta}\right) \tag{4.24}$$

for large $|\zeta|$ and

$$\theta_0(\zeta) = \frac{-k}{\rho\zeta}\left[\overline{h_0} - \frac{\overline{l_0}}{(\overline{m_\tau}(1/\zeta))^2}\right] \overline{m'_\tau}\left(\frac{1}{\zeta}\right) + \mathrm{O}(\zeta) = \frac{-k}{\rho} \overline{a}_{1\tau} \overline{h_0} \frac{1}{\zeta} + \mathrm{O}(\zeta) \tag{4.25}$$

for small $|\zeta|$.

By combining Eqs. (4.24) and (4.25), we may obtain the following result

$$\theta_0(\zeta) = \frac{k\zeta}{\rho} g''(\zeta) = \frac{k}{\rho}\left[a_{1\tau} h_0 \zeta - \overline{a_{1\tau}}\overline{h_0}\zeta^{-1}\right] \tag{4.26}$$

where a temperature function $g(\zeta)$ can then be determined by integrating Eq. (4.26) twice with respect to ζ and z_τ as

$$g(\zeta) = \tfrac{1}{2} h_0 a_{1\tau}^2 \zeta^2 + \left(a_{1\tau}\overline{a_{1\tau}}\overline{h_0} - a_{1\tau} a_{2\tau} h_0\right)\log\zeta + \tfrac{1}{2}\overline{h_0}\overline{a_{1\tau}} a_{2\tau}\zeta^{-2} \tag{4.27}$$

and comparing Eq. (4.27) with Eq. (3.19), the constant l_0 can be found as

$$l_0 = a_{1\tau}\overline{a_{1\tau}}\overline{h_0} - a_{1\tau} a_{2\tau} h_0 \tag{4.28}$$

Since the entire boundary of the elliptic hole is traction-free, the solution given in Eq. (3.33) reduces to

$$\mathbf{\Theta}'(\zeta) = \mathbf{p}(\zeta) \tag{4.29}$$

Substituting Eqs. (4.1) and (4.2) into Eqs. (3.46) and (3.47) with knowing that $\mathbf{q} = 0$ and $\mathbf{h}(\zeta) = 0$, the properties of $\mathbf{\Theta}'(\zeta)$ at infinity and origin, respectively are found as

$$\mathbf{\Theta}'(\zeta) = \mathbf{Br} + \mathbf{d} l_0 + \mathrm{O}(\zeta^{-1}), \quad |\zeta| \to \infty \tag{4.30}$$

and

$$\mathbf{\Theta}'(\zeta) = \overline{\mathbf{B}}\overline{\mathbf{r}} + \overline{\mathbf{d}}\,\overline{l_0} + \mathrm{O}(\zeta), \quad |\zeta| \to 0 \tag{4.31}$$

where $\mathbf{r} = -\mathbf{A}^{\mathrm{T}}(\mathbf{d} l_0 - \overline{\mathbf{d}}\,\overline{l_0}) - \mathbf{B}^{\mathrm{T}}(\mathbf{c} l_0 - \overline{\mathbf{c}}\overline{l_0})$.

Notice that during the derivation of Eqs. (4.30) and (4.31) the complex functions $\mathbf{d} h_0 m_\tau(\zeta)$ and $\overline{\mathbf{d}}\,\overline{h_0}\overline{m_\tau}(1/\zeta)$ associated with the solutions of the homogeneous problem due to a uniform heat flow, which will not produce stress, have been subtracted from Eqs. (3.46) and (3.47), respectively. By combining Eqs. (4.29)–(4.31), the final expression for $\mathbf{\Theta}'(\zeta)$ can be obtained as

$$\mathbf{\Theta}'(\zeta) = \mathbf{Br} + \mathbf{d} l_0 \tag{4.32}$$

where $\mathbf{Br} + \mathbf{d} l_0$ is a real vector. Moreover, the temperature function can be obtained as

$$g(\zeta) = \left(a_{1\tau}\overline{a_{1\tau}}\overline{h_0} - a_{1\tau}a_{2\tau}h_0\right)\log\zeta + \tfrac{1}{2}\left(\overline{h}_0\overline{a_{1\tau}} - h_0 a_{2\tau}\right)a_{2\tau}\zeta^{-2} \tag{4.33}$$

The solution of $\mathbf{f}(\zeta)$ can then be found by applying the first term of Eqs. (3.28) and (4.33) to Eq. (4.32) and integrating the results with respect to ζ as

$$\mathbf{f}(\zeta) = \tfrac{1}{2}\mathbf{B}^{-1}\mathbf{d}\left(h_0 a_{2\tau} - \overline{h_0 a_{1\tau}}\right)a_{2\tau}\zeta^{-2} + \mathbf{r}\log\zeta \tag{4.34}$$

which is proved to agree with the one obtained earlier by Chao and Shen (1998).

5. A partially reinforced elliptic hole under remote heat flow

In this section, we consider a partially reinforced elliptic hole embedded in an infinite anisotropic medium under a remote uniform heat flow (see Fig. 2). Suppose that the hole is subjected to a uniform heat flow q_0 with an angle γ_0 measured from the x_1-axis and is isolated on the whole boundary including the reinforced segment between $(a\cos\varphi, -b\sin\varphi)$ and $(a\cos\varphi, b\sin\varphi)$ which is mapped onto arc $L = (e^{-i\varphi}, e^{i\varphi})$ in the ζ-plane. We know that the reinforced inclusion produces a rotation under uniform heat flow, therefore the displacement vector of the reinforced inclusion can be found as (Ting, 1996)

$$\hat{\mathbf{u}} = \varepsilon^*[a\cos\varphi\mathbf{m}(0) - b\sin\varphi\mathbf{n}(0)] = \varepsilon^*\mathrm{Re}\left[-ie^{-i\varphi}\mathbf{y}\right] \tag{5.1}$$

where $\mathbf{y} = b\mathbf{n}(0) + ia\mathbf{m}(0)$ and ε^* is a rotation angle.

The displacement gradient along the tangent direction $\mathbf{n}$ of the hole boundary can be obtained as

$$\mathbf{u}_{,n} = \frac{\partial\hat{\mathbf{u}}}{\partial n} = \frac{\varepsilon^*}{2\rho}\left(\frac{\mathbf{y}}{\sigma} + \sigma\overline{\mathbf{y}}\right) \tag{5.2}$$

and the temperature function $g'(\zeta)$ can be determined by differentiating Eq. (4.33) with respect to ζ as

$$g'(\zeta) = \left(h_0 a_{2\tau} - \overline{h}_0\overline{a_{1\tau}}\right)a_{2\tau}\zeta^{-3} - \left(h_0 a_{2\tau} - \overline{h}_0\overline{a_{1\tau}}\right)a_{1\tau}\zeta^{-1} \tag{5.3}$$

With the results of Eqs. (5.2) and (5.3) and the properties of $\mathbf{\Theta}'$ at infinity and origin given in Eqs. (4.30) and (4.31), the final expression of the stress function $\mathbf{\Theta}'$, similar to the previous case, can be obtained as

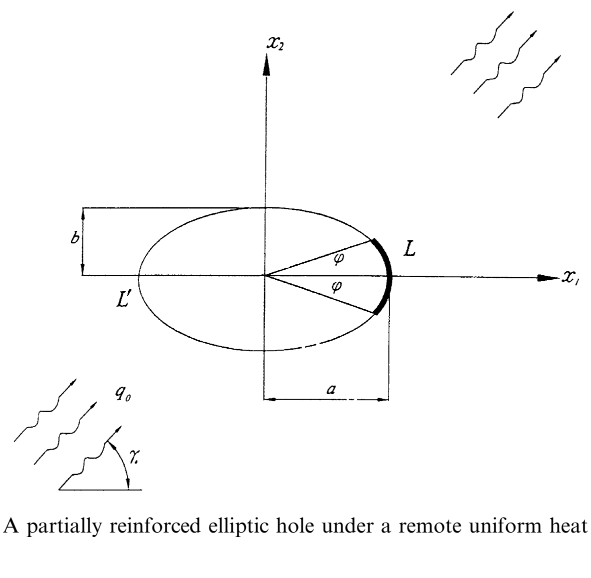

Fig. 2. A partially reinforced elliptic hole under a remote uniform heat flow q_0.

$$\mathbf{\Theta}'(\zeta) = -\left(\mathbf{I}+\mathbf{M}\overline{\mathbf{M}}^{-1}\right)^{-1}\mathbf{M}\left\{\frac{\varepsilon^*}{2}\left(\frac{\mathbf{y}}{\zeta}+\zeta\overline{\mathbf{y}}\right)+\mathrm{i}\left[\left(\mathbf{c}-\mathbf{A}\mathbf{B}^{-1}\mathbf{d}\right)\left[\left(h_0a_{2\tau}-\overline{h_0}\overline{a_{1\tau}}\right)a_{2\tau}\zeta^{-2}-\left(h_0a_{2\tau}-\overline{h_0}\overline{a_{1\tau}}\right)a_{1\tau}\right]\right.\right.$$
$$\left.\left.-\left(\overline{\mathbf{c}}-\overline{\mathbf{A}}\overline{\mathbf{B}}^{-1}\overline{\mathbf{d}}\right)\left[\left(\overline{h_0}\overline{a_{2\tau}}-h_0a_{1\tau}\right)\overline{a_{2\tau}}\zeta^2-\left(\overline{h_0}\overline{a_{2\tau}}-h_0a_{1\tau}\right)\overline{a_{1\tau}}\right]\right]\right\}+\mathbf{X}(\zeta)\mathbf{p}(\zeta)$$

where $\mathbf{X}(\zeta)$ is given in Eq. (4.13) and the polynomial vector $\mathbf{p}(\zeta)$ is defined as

$$\mathbf{p}(\zeta) = \mathbf{e}_3\zeta^3+\mathbf{e}_2\zeta^2+\mathbf{e}_1\zeta+\mathbf{e}_0+\mathbf{e}_{-1}\zeta^{-1}+\mathbf{e}_{-2}\zeta^{-2} \tag{5.5}$$

The constant coefficients in Eq. (5.5) satisfy

$$\mathbf{e}_3 = -\mathrm{i}\mathbf{\Lambda}^{-1}\left(\mathbf{I}+\mathbf{M}\overline{\mathbf{M}}^{-1}\right)^{-1}\mathbf{M}\left(\overline{\mathbf{c}}-\overline{\mathbf{A}}\overline{\mathbf{B}}^{-1}\overline{\mathbf{d}}\right)\left(\overline{h_0}\overline{a_{2\tau}}-h_0a_{1\tau}\right)\overline{a_{2\tau}}$$

$$\mathbf{e}_2 = \mathbf{\Lambda}^{-1}\left(\mathbf{I}+\mathbf{M}\overline{\mathbf{M}}^{-1}\right)^{-1}\mathbf{M}\frac{\varepsilon^*}{2}\overline{\mathbf{y}}-\mathbf{G}_1\mathbf{e}_3$$

$$\mathbf{e}_1 = \mathbf{\Lambda}^{-1}\left\{(\mathbf{Br}+\mathbf{d}l_0)-\mathrm{i}\left(\mathbf{I}+\mathbf{M}\overline{\mathbf{M}}^{-1}\right)^{-1}\mathbf{M}\left[\left(\mathbf{c}-\mathbf{A}\mathbf{B}^{-1}\mathbf{d}\right)\left(h_0a_{2\tau}-\overline{h_0}\overline{a_{1\tau}}\right)a_{1\tau}-\left(\overline{\mathbf{c}}-\overline{\mathbf{A}}\overline{\mathbf{B}}^{-1}\overline{\mathbf{d}}\right)\left(\overline{h_0}\overline{a_{2\tau}}\right.\right.\right.$$
$$\left.\left.\left.-h_0a_{1\tau}\right)\overline{a_{1\tau}}\right]\right\}-\mathbf{G}_1\mathbf{e}_2-\mathbf{G}_2\mathbf{e}_3$$

$$\mathbf{e}_{-2} = \mathrm{i}\langle\langle -\mathrm{e}^{2\varphi\varepsilon_\alpha}\rangle\rangle\mathbf{\Lambda}^{-1}\left(\mathbf{I}+\mathbf{M}\overline{\mathbf{M}}^{-1}\right)^{-1}\mathbf{M}\left(\mathbf{c}-\mathbf{A}\mathbf{B}^{-1}\mathbf{d}\right)\left(h_0a_{2\tau}-\overline{h_0}\overline{a_{1\tau}}\right)a_{2\tau}$$

$$\mathbf{e}_{-1} = \langle\langle -\mathrm{e}^{2\varphi\varepsilon_\alpha}\rangle\rangle\mathbf{\Lambda}^{-1}\left(\mathbf{I}+\mathbf{M}\overline{\mathbf{M}}^{-1}\right)^{-1}\mathbf{M}\frac{\varepsilon^*}{2}\mathbf{y}-\mathbf{G}_4\mathbf{e}_{-2}$$

$$\mathbf{e}_0 = \langle\langle -\mathrm{e}^{2\varphi\varepsilon_\alpha}\rangle\rangle\mathbf{\Lambda}^{-1}\left\{(\mathbf{Br}+\mathbf{d}l_0)-i\left(\mathbf{I}+\mathbf{M}\overline{\mathbf{M}}^{-1}\right)^{-1}\mathbf{M}\left[\left(\mathbf{c}-\mathbf{A}\mathbf{B}^{-1}\mathbf{d}\right)\left(h_0a_{2\tau}-\overline{h_0}\overline{a_{1\tau}}\right)a_{1\tau}\right.\right.$$
$$\left.\left.-\left(\overline{\mathbf{c}}-\overline{\mathbf{A}}\overline{\mathbf{B}}^{-1}\overline{\mathbf{d}}\right)\left(\overline{h_0}\overline{a_{2\tau}}-h_0a_{1\tau}\right)\overline{a_{1\tau}}\right]\right\}-\mathbf{G}_4\mathbf{e}_{-1}-\mathbf{G}_5\mathbf{e}_{-2} \tag{5.6}$$

where $\mathbf{r} = -\mathbf{A}^{\mathrm{T}}(\mathbf{d}l_0-\overline{\mathbf{d}}\,\overline{l_0})-\mathbf{B}^{\mathrm{T}}(\mathbf{c}l_0-\overline{\mathbf{c}}\overline{l_0})$ as found from Eq. (3.44) by letting $\hat{\mathbf{p}} = 0$.

As for the rotation angle ε^*, we impose the condition that the resultant moment about the x_3-axis due to the traction $\mathbf{t}_m$ on the elliptic boundary vanishes, i.e.,

$$\int_L [a\cos\varphi\mathbf{m}(0)-b\sin\varphi\mathbf{n}(0)]\mathbf{t}_m\rho\,\mathrm{d}\varphi = 0 \tag{5.7}$$

With the help of Eqs. (3.29) and (3.9), Eq. (5.7) can be replaced by

$$\int_L \frac{\mathrm{i}}{2}(\sigma^{-2}\mathbf{y}-\overline{\mathbf{y}})\left[\mathbf{\Theta}'(\sigma^+)-\mathbf{\Theta}'(\sigma^-)\right]\mathrm{d}\sigma = 0 \tag{5.8}$$

and substituting Eq. (5.4) into Eq. (5.8) and applying residue theory, we have a rotation angle as

$$\varepsilon^* = \frac{\mathbf{y}^{\mathrm{T}}\mathbf{\Lambda}\langle\langle\mathrm{e}^{-2\varphi\varepsilon_\alpha}\rangle\rangle\mathbf{S}_1-\overline{\mathbf{y}}^{\mathrm{T}}\mathbf{\Lambda}\mathbf{S}_2}{\mathbf{S}_3+\mathbf{S}_4+\mathbf{S}_5} \tag{5.9}$$

where the constant vectors in Eq. (5.9) are expressed as

$$\begin{aligned}
\mathbf{S}_1 &= \mathbf{Y}_1 - (\mathbf{G}_2 - \mathbf{G}_1^2)\mathbf{e}_3 + \mathbf{G}_4\mathbf{Y}_2 - (2\mathbf{G}_4\mathbf{G}_5 - \mathbf{G}_4^3 - \mathbf{G}_6)\mathbf{e}_{-2} \\
\mathbf{S}_2 &= \mathbf{Y}_2 - (\mathbf{G}_5 - \mathbf{G}_4^2)\mathbf{e}_{-2} + \mathbf{G}_1\mathbf{Y}_1 - (2\mathbf{G}_1\mathbf{G}_2 - \mathbf{G}_1^3 - \mathbf{G}_3)\mathbf{e}_3 \\
\mathbf{S}_3 &= \mathbf{y}^{\mathrm{T}}\mathbf{\Lambda}\langle\langle \mathrm{e}^{-2\varphi\varepsilon_\alpha}\rangle\rangle\mathbf{G}_1\mathbf{R}\overline{\mathbf{y}} \\
\mathbf{S}_4 &= \overline{\mathbf{y}}^{\mathrm{T}}\mathbf{\Lambda}\langle\langle \mathrm{e}^{2\varphi\varepsilon_\alpha}\rangle\rangle\mathbf{G}_4\mathbf{R}\mathbf{y} \\
\mathbf{S}_5 &= \frac{\mathbf{y}^{\mathrm{T}}}{2}\mathbf{\Lambda}(\mathbf{G}_5 - \mathbf{G}_4^2)\mathbf{R}\mathbf{y} + \frac{\overline{\mathbf{y}}^{\mathrm{T}}}{2}\mathbf{\Lambda}(\mathbf{G}_2 - \mathbf{G}_1^2)\mathbf{R}\overline{\mathbf{y}} - \mathbf{y}^{\mathrm{T}}\mathbf{\Lambda}\mathbf{R}\overline{\mathbf{y}}
\end{aligned} \tag{5.10}$$

and

$$\begin{aligned}
\mathbf{Y}_1 &= \mathbf{\Lambda}^{-1}\Big\{(\mathbf{B}\mathbf{r} + \mathbf{d}l_0) - \mathrm{i}\Big(\mathbf{I} + \mathbf{M}\overline{\mathbf{M}}^{-1}\Big)^{-1}\mathbf{M}\Big[\big(\mathbf{c} - \mathbf{A}\mathbf{B}^{-1}\mathbf{d}\big)\big(h_0 a_{2\tau} - \overline{h_0}\overline{a_{1\tau}}\big)a_{1\tau} \\
&\qquad - \Big(\overline{\mathbf{c}} - \overline{\mathbf{A}}\overline{\mathbf{B}}^{-1}\overline{\mathbf{d}}\Big)\Big(\overline{h_0}\overline{a_{2\tau}} - h_0 a_{1\tau}\Big)\Big]\Big\} \\
\mathbf{Y}_2 &= \langle\langle -\mathrm{e}^{2\varphi\varepsilon_\alpha}\rangle\rangle\mathbf{Y}_1 \\
\mathbf{R} &= \mathbf{\Lambda}^{-1}\Big(\mathbf{I} + \mathbf{M}\overline{\mathbf{M}}^{-1}\Big)^{-1}\mathbf{M}
\end{aligned} \tag{5.11}$$

Now we have completed the solution for the problem with a partially reinforced elliptic hole under a remote uniform heat flow. When the reinforced segment is assumed to vanish, the solution given in Eq. (5.4) can be reduced to the one given in Eq. (4.29) corresponding to an infinite plate with an elliptic hole subjected to a uniform heat flow at infinity (Chao and Shen, 1998).

6. Illustrative examples

In order to demonstrate the use of the present approach and to understand clearly the physical behavior of the mixed boundary-value problems, numerical examples associated with an elliptic hole boundary under indentation and a partially reinforced elliptic hole under a remote uniform heat flow will be discussed in this section.

6.1. A rigid stamp on an elliptic hole boundary

We first consider that an infinite body containing an elliptic hole with $b/a = 0.6$ is subjected to a resultant heat flow $\hat{Q}$ approached from the negative x_1-axis and is loaded by a rigid stamp with a resultant force $\hat{\mathbf{p}} = (\hat{p}, 0, 0)$ along the segment $\varphi = 30°$. The material properties considered in the present study are chosen as

$$\begin{aligned}
&E_{11} = 144.8\ \text{Gpa}, \quad E_{22} = E_{33} = 9.7\ \text{Gpa}, \quad \nu_{12} = \nu_{23} = \nu_{13} = 0.3 \\
&G_{12} = G_{23} = G_{13} = 4.1\ \text{Gpa}, \quad k_{11} = 4.62\ \mathrm{W\,m^{-1}\,K^{-1}}, \quad k_{22} = k_{33} = 0.72\ \mathrm{W\,m^{-1}\,K^{-1}} \\
&\alpha_{11} = -0.3 \times 10^{-6}\ \mathrm{K^{-1}}, \quad \alpha_{22} = \alpha_{33} = 28.1 \times 10^{-6}\ \mathrm{K^{-1}}
\end{aligned} \tag{6.1}$$

Based on the material properties listed above, the thermal eigenvalue τ and elasticity eigenvalues p_α can be determined from Eqs. (2.7) and (2.12), respectively. The matrices $\mathbf{\Lambda}$ and $\mathbf{\Gamma}(\zeta)$ can also be found from Eqs. (3.15) and (3.36) as the stamp location is given. It is found that, based on the material properties given in Eq. (6.1), the eigenvalues $\delta_\alpha(\alpha = 1, 2, 3)$ corresponding to Eq. (3.36) are $\delta_1 = -0.5 - 0.0863\mathrm{i}$, $\delta_2 = -0.5 + 0.0863\mathrm{i}$, $\delta_3 = -0.5$ which implies that the stresses would change sign near the ends of the contact portion. However, the oscillatory zone at such locations is extremely small for the case of a rigid stamp. By using the transformation (4.1) and (4.2), the general solution of $\theta_0(\zeta)$ and $\mathbf{\Theta}'(\zeta)$ can be found from Eqs. (4.10) and (4.12), respectively. Consequently, the function $g(\zeta)$ and $\mathbf{f}(\zeta)$, respectively can be determined from Eqs. (3.12) and (3.28), and the stress function $\phi(\zeta)$ is then obtained from Eq. (3.4). The contact stress σ_{mm} under the rigid stamp is related to the stress function $\phi(\zeta)$ by

$$\sigma_{mm} = m^{\mathrm{T}}(\theta)\phi_{,n} \tag{6.2}$$

Notice that during the calculation of contact stress defined in Eq. (6.2), a replacement of $\zeta_\tau, \zeta_1, \zeta_2, \zeta_3$ has been made for each function stated above. For the purpose of clearly expressing the effect of material properties, geometric configuration and applied loading on the contact stress, the nondimensional parameter λ^* defined as

$$\lambda^* = \frac{\alpha_{11}E_{11}\hat{Q}b\sin\varphi}{k_{11}\hat{p}} \tag{6.3}$$

is used which must be properly chosen such that the condition of a negative (compressive) contact stress should be satisfied. In general, λ^* ranges from $-\infty$ to ∞ which describes the indentation problem under both thermal and mechanical loading conditions. In the present case, the perfect contact is found to maintain throughout the punch face as λ^* ranges from –0.882 to 0.067. For $\lambda^* \geqslant 0.067$, corresponding to a sufficiently large heat flux into the infinite body from the rigid punch, tensile contact stresses are predicted near the ends of the rigid stamp. For $\lambda^* \leqslant -0.882$, tensile contact stresses are predicted near the middle of the rigid stamp which will result in imperfect contact. The nondimensionalized contact stress $\sigma_{mm}/(\hat{p}/2b\sin\varphi)$ with $\lambda^* = -0.5$ and 0 is shown in Figs. 3 and 4, respectively which indicates that the stress singularity is found near the ends of the rigid stamp. Notice that the oscillatory behavior near the ends of

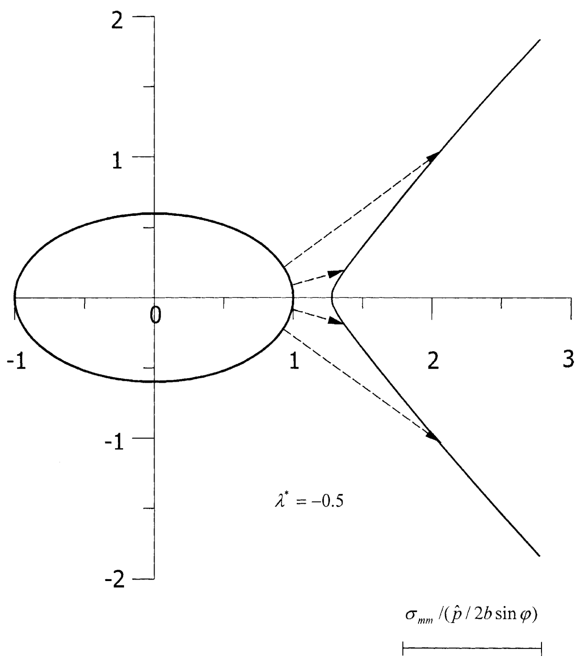

Fig. 3. The nondimensionalized contact stress $\sigma_{mm}/(\hat{p}/2b\sin\varphi)$ along the elliptic hole indented by a rigid stamp with $\lambda^* = -0.5$.

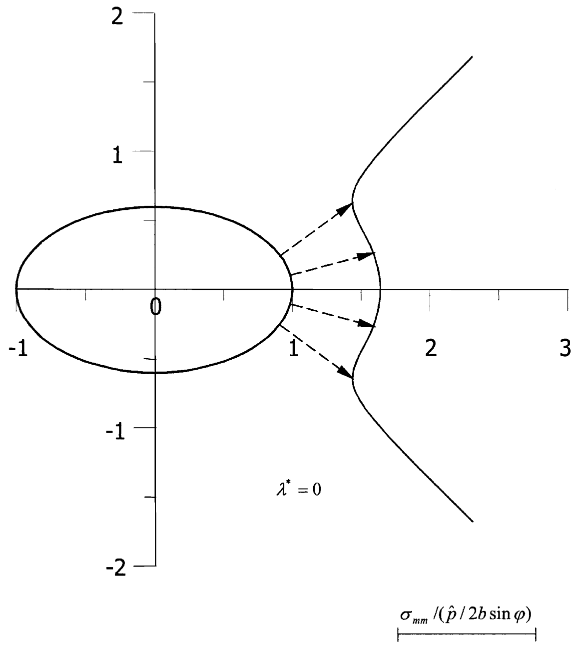

Fig. 4. The nondimensionalized contact stress $\sigma_{mm}/(\hat{p}/2b\sin\varphi)$ along the elliptic hole indented by a rigid stamp with $\lambda^* = 0$.

the rigid stamp cannot be seen from these figures since the oscillatory zone is extremely small as discussed in the last paragraph.

6.2. A partially reinforced elliptic hole under remote heat flow

As a second example, we consider an infinite body containing an elliptic hole with a partially reinforced segment $\varphi = 30°$ under a remote uniform heat flow (see Fig. 2). The material properties used in this example are listed in Eq. (6.1). With this specification, the stress function $\phi(\zeta)$ can be obtained from Eq. (3.4) with the help of substituting Eqs. (5.3)–(5.6) into Eq. (3.28). Since the transformation function is single-valued along the elliptic hole boundary, the exact solution of the radial stress $\sigma_{mm}k_{11}/\alpha_{22}q_0G_{12}a$ along the reinforced segment can be found by applying Eq. (6.2). The results shown in Fig. 5 ($\gamma_0 = 0°$) and Fig. 6 ($\gamma_0 = 30°$) reveal that the stress becomes unbounded at the ends of the reinforced portion. The oscillatory behavior at the ends of the reinforced portion cannot be seen from Fig. 5 or Fig. 6 since the oscillatory zone at such locations is extremely small.

7. Conclusions

A general solution for the mixed boundary-value problems of two-dimensional anisotropic thermoelasticity is obtained by employing the Stroh formalism, the method of analytical continuation, and the technique of mapping an elliptic curve to a unit circle. Through these general solutions, two typical examples are fully discussed. Because the transformation function used in our solutions is always single-valued, we can obtain the exact solutions in the derivations. Notice that the solution derived in Sections 4

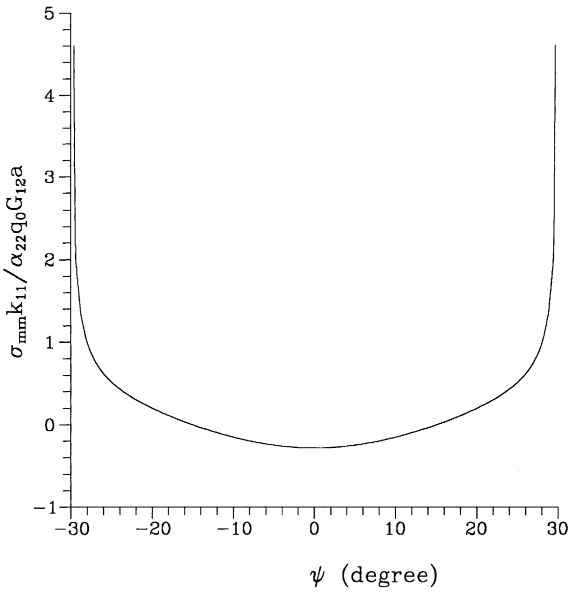

Fig. 5. The nondimensionalized radial stress $\sigma_{mm}k_{11}/\alpha_{22}q_0G_{12}a$ along the reinforced segment with $\gamma_0 = 0°$.

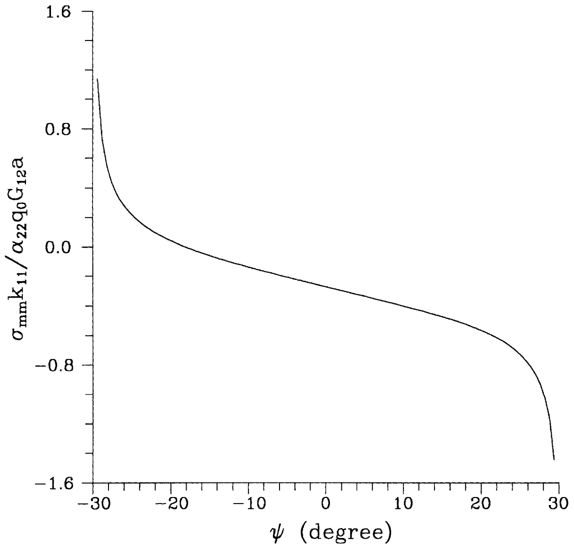

Fig. 6. The nondimensionalized radial stress $\sigma_{mm}k_{11}/\alpha_{22}q_0G_{12}a$ along the reinforced segment with $\gamma_0 = 30°$.

and 5 are valid for the entire full field only when a replacement of $\zeta_\tau, \zeta_1, \zeta_2, \zeta_3$ should be made for each component function to calculate the displacement and stress from Eqs. (2.8) and (2.9). This observation, which was pointed out by Suo (1990), makes the analytic function continuation possible. The mixed boundary-value problems associated with a point heat source can also be derived in a similar fashion by choosing the proper expression of $H(z_\alpha)$ in Eq. (3.37) such that the existence of single-valued displacements in the entire system is guaranteed.

Appendix A

The external applied force $\hat{\mathbf{p}}$ can be expressed as

$$\begin{aligned}\hat{\mathbf{p}} &= \int_L \mathbf{t}_m \, \mathrm{d}n = -\int_L \frac{\mathrm{i}}{\rho}\left[\mathbf{\Theta}'(\sigma^+) - \mathbf{\Theta}'(\sigma^-)\right] \mathrm{d}n = \int_L \frac{\mathbf{\Theta}'(\sigma^+) - \mathbf{\Theta}'(\sigma^-)}{\sigma} \, \mathrm{d}\sigma \\ &= \frac{1}{2\pi \mathrm{i}} \mathbf{\Lambda} \langle\langle 1 + \mathrm{e}^{-2\pi\varepsilon_\alpha} \rangle\rangle \int_L \frac{\mathbf{\Gamma}^+(\sigma)}{\sigma} \int_L \frac{1}{t-\sigma} [\mathbf{\Gamma}^+(t)]^{-1} \mathbf{\Lambda}^{-1} \mathbf{M} 2\,\mathrm{Im}\left[(\mathbf{c} - \mathbf{A}\mathbf{B}^{-1}\mathbf{d}) t g'(t)\right] \mathrm{d}t \, \mathrm{d}\sigma \\ &\quad + \mathbf{\Lambda} \int_L \frac{\mathbf{\Gamma}^+(\sigma) - \mathbf{\Gamma}^-(\sigma)}{\sigma} \mathbf{p}(\sigma) \, \mathrm{d}\sigma \end{aligned} \tag{A.1}$$

By changing the order of integration, the integral term in Eq. (A.1) can be rewritten as

$$\int_L [\mathbf{\Gamma}^+(t)]^{-1} \left\{ 2\mathbf{\Lambda}^{-1} \mathbf{M}\,\mathrm{Im}\left[(\mathbf{c} - \mathbf{A}\mathbf{B}^{-1}\mathbf{d}) t g'(t)\right] \right\} \int_L \frac{\mathbf{\Gamma}^+(\sigma)}{\sigma(t-\sigma)} \, \mathrm{d}\sigma \, \mathrm{d}t \tag{A.2}$$

which is in fact zero since the integral term $\int_L (\mathbf{\Gamma}^+(\sigma)/\sigma(t-\sigma)) \, \mathrm{d}\sigma$ is found to vanish. Thus, Eq. (A.1) can be reduced to

$$\hat{\mathbf{p}} = \int_L \mathbf{t}_m \mathrm{d}n = \int_L \frac{\mathbf{\Theta}'(\sigma^+) - \mathbf{\Theta}'(\sigma^-)}{\sigma} \, \mathrm{d}\sigma = \mathbf{\Lambda} \int_L \frac{\mathbf{\Gamma}^+(\sigma) - \mathbf{\Gamma}^-(\sigma)}{\sigma} \mathbf{p}(\sigma) \, \mathrm{d}\sigma \tag{A.3}$$

The integral terms in Eq. (A.3) can be expressed as

$$\begin{aligned}\oint_{\mathrm{c}} \frac{\mathbf{\Theta}'(\zeta)}{\zeta} \, \mathrm{d}\zeta &= \int_{\mathrm{c}_1} \frac{\mathbf{\Theta}'(\sigma^+)}{\sigma} \, \mathrm{d}\sigma + \int_{\mathrm{c}_2} \frac{\mathbf{\Theta}'(\sigma^-)}{\sigma} \, \mathrm{d}\sigma + \int_{\mathrm{c}_0} \frac{\mathbf{\Theta}'(\zeta)}{\zeta} \, \mathrm{d}\zeta + \int_{\mathrm{c}_\infty} \frac{\mathbf{\Theta}'(\zeta)}{\zeta} \, \mathrm{d}\zeta \\ &= \int_L \frac{\mathbf{\Theta}'(\sigma^+)}{\sigma} \, \mathrm{d}\sigma - \int_L \frac{\mathbf{\Theta}'(\sigma^-)}{\sigma} \, \mathrm{d}\sigma + \int_{\mathrm{c}_0} \frac{\mathbf{\Theta}'(\zeta)}{\zeta} \, \mathrm{d}\zeta + \int_{\mathrm{c}_\infty} \frac{\mathbf{\Theta}'(\zeta)}{\zeta} \, \mathrm{d}\zeta = 0 \end{aligned} \tag{A.4}$$

and

$$\begin{aligned}\mathbf{\Lambda} \oint_{\mathrm{c}} \frac{\mathbf{\Gamma}(\zeta)\mathbf{p}(\zeta)}{\zeta} \, \mathrm{d}\zeta &= \mathbf{\Lambda} \left\{ \int_{\mathrm{c}_1} \frac{\mathbf{\Gamma}(\sigma^+)\mathbf{p}(\sigma)}{\sigma} \, \mathrm{d}\sigma + \int_{\mathrm{c}_1} \frac{\mathbf{\Gamma}(\sigma^-)\mathbf{p}(\sigma)}{\sigma} \, \mathrm{d}\sigma + \int_{\mathrm{c}_0} \frac{\mathbf{\Gamma}(\zeta)\mathbf{p}(\zeta)}{\zeta} \, \mathrm{d}\zeta \right. \\ &\qquad \left. + \int_{\mathrm{c}_\infty} \frac{\mathbf{\Gamma}(\zeta)\mathbf{p}(\zeta)}{\zeta} \, \mathrm{d}\zeta \right\} \\ &= \mathbf{\Lambda} \left\{ \int_L \frac{\mathbf{\Gamma}(\sigma^+)\mathbf{p}(\sigma)}{\sigma} \, \mathrm{d}\sigma - \int_L \frac{\mathbf{\Gamma}(\sigma^-)\mathbf{p}(\sigma)}{\sigma} \mathrm{d}\sigma + \int_{\mathrm{c}_0} \frac{\mathbf{\Gamma}(\zeta)\mathbf{p}(\zeta)}{\zeta} \, \mathrm{d}\zeta \right. \\ &\qquad \left. + \int_{\mathrm{c}_\infty} \frac{\mathbf{\Gamma}(\zeta)\mathbf{p}(\zeta)}{\zeta} \, \mathrm{d}\zeta \right\} = 0 \end{aligned} \tag{A.5}$$

where the integration contour is shown in Fig. 7. On substitution of Eqs. (A.4) and (A.5) into Eq. (A.3) we have the following relations

$$\int_{\mathrm{c}_0} \frac{\mathbf{\Theta}'(\zeta)}{\zeta} \, \mathrm{d}\zeta = \mathbf{\Lambda} \int_{\mathrm{c}_0} \frac{\mathbf{\Gamma}(\zeta)\mathbf{p}(\zeta)}{\zeta} \, \mathrm{d}\zeta \tag{A.6}$$

and

$$\int_{\mathrm{c}_\infty} \frac{\mathbf{\Theta}'(\zeta)}{\zeta} \, \mathrm{d}\zeta = \mathbf{\Lambda} \int_{\mathrm{c}_\infty} \frac{\mathbf{\Gamma}(\zeta)\mathbf{p}(\zeta)}{\zeta} \, \mathrm{d}\zeta \tag{A.7}$$

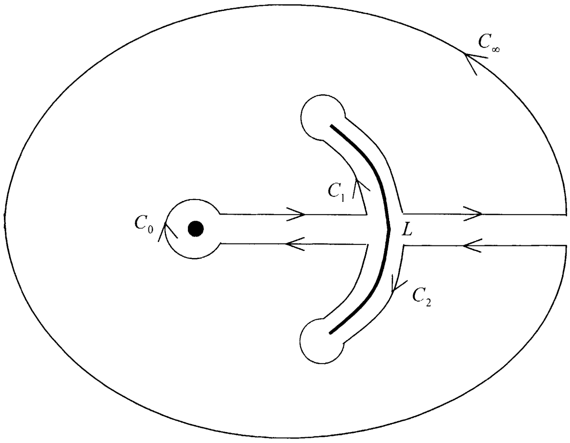

Fig. 7. The integration contour for Eqs. (A.4) and (A.5).

Once the conditions of $\mathbf{\Theta}'(\zeta)$ and $\mathbf{\Gamma}(\zeta)$ at infinity and at the origin are given, the coefficients in $\mathbf{p}(\zeta)$ can be determined by solving Eqs. (A.6) and (A.7) in a straightforward manner.

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